

Technology and cost minimization

ECON 201 - Winter 2026

Week 1: January 7th

Sasha Petrov

Logistics

Lectures and office hours

- Materials in the lectures should be enough for passing the class
- Slides and optional notes will be on Canvas
- Office hours:
 - Will set up a survey to find the best time for everyone
 - Both group and individual sessions

Production technology

Today's Agenda

1. Production functions
2. Marginal rate of technical substitution
3. Returns to scale

Firm's problem

A firm chooses the level of output q and the composition of inputs z to maximize its profits π :

$$\max_{q,z} \quad \pi \equiv pq - c$$

$$\text{s.t.} \quad p = p^*$$

$$q = f(z)$$

$$c = w^* \cdot z$$

- q is the output quantity
- z are the inputs (factors of production)
- f is the production function
- p^* is the market price for the produced good
- w^* are the market prices for the inputs

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$$\begin{aligned} \max_{q,z} \quad & \pi \equiv pq - c \\ \text{s.t.} \quad & p = p^* && \text{perfect competition} \\ & q = f(z) && \text{technological constraint} \\ & c = w^* \cdot z && \text{cost of production} \end{aligned}$$

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Production Technology

Production Function

Production Function: A function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that describes the maximum output that can be produced from a given combination of inputs.

In case $z \equiv (K, L)$:

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Two important characteristics of f :

- Technical rate of substitution
 - Determines the composition of sourced inputs
- Returns to scale
 - Determines the optimal size of firms

Marginal Products

The **marginal product** of an input is the additional output from one more unit of that input, holding other inputs constant.

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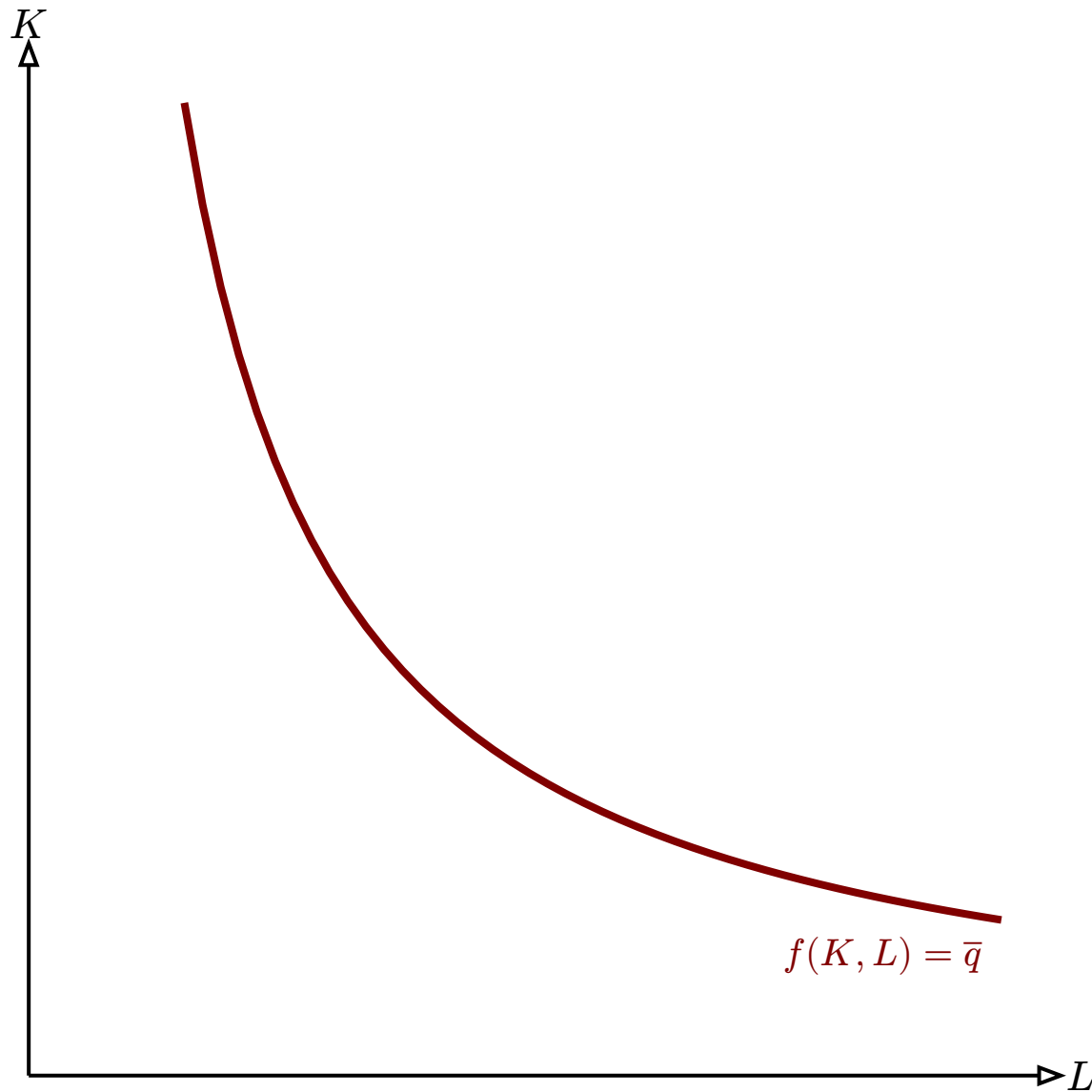
The **marginal rate of technical substitution** is:

$$\text{MRTS} = \frac{\partial f}{\partial L} \bigg/ \frac{\partial f}{\partial K}$$

How substitutable are the inputs in the production process?

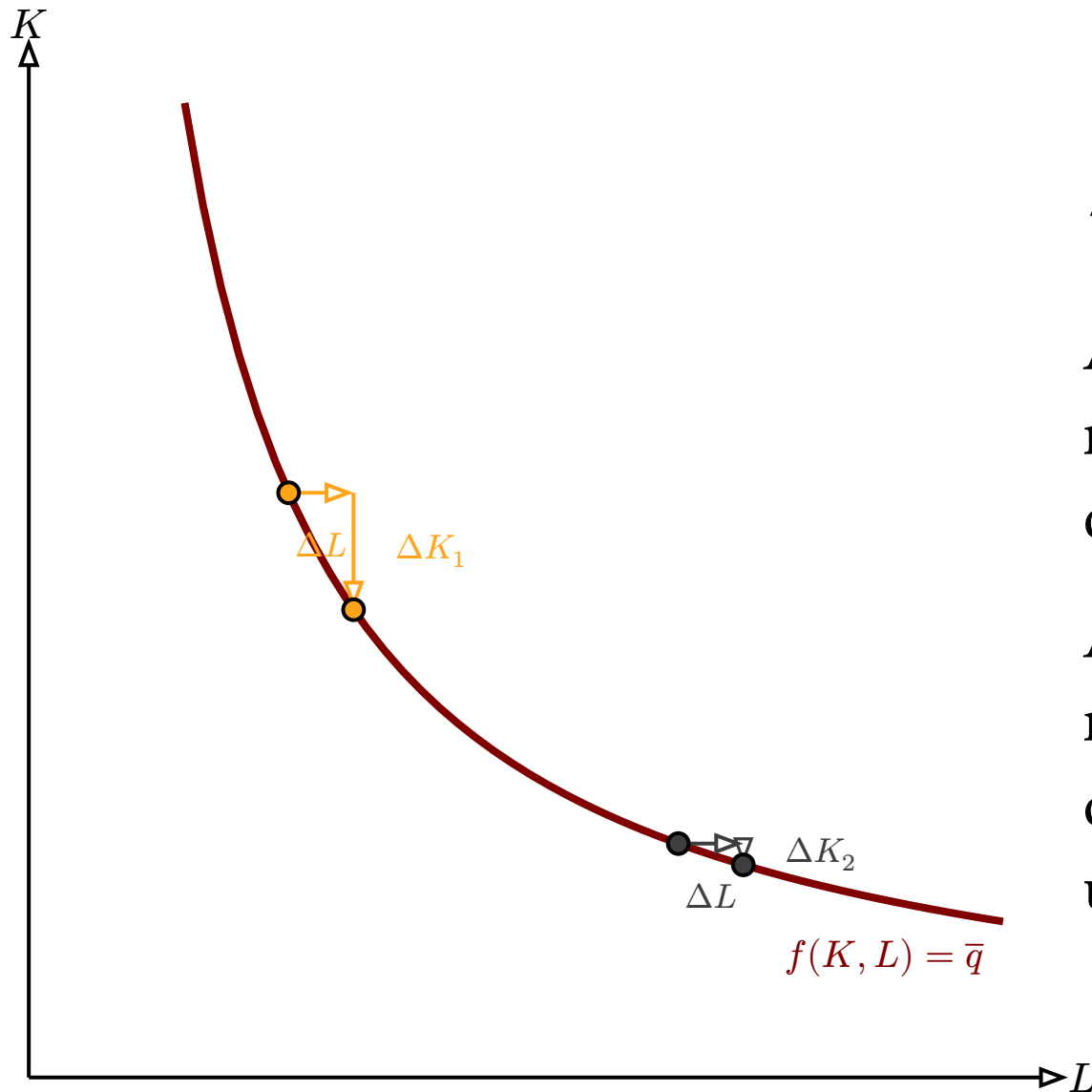
Determines the response of the input composition to price changes.

Marginal Products



Cobb-Douglas Isoquant: $q = K^{0.5} L^{0.5} = 4$

Marginal Products



Two interpretations:

As L increases, the marginal 'savings' of K go down.

As L decreases, the necessary marginal compensation by K goes up.

Cobb-Douglas Isoquant: $q = K^{0.5}L^{0.5} = 4$

Returns to Scale

A production function exhibits:

- **Constant returns to scale (CRS)** if:

$$f(tK, tL) = t \cdot f(K, L) \quad \forall t > 0$$

- **Increasing returns to scale (IRS)** if:

$$f(tK, tL) > t \cdot f(K, L) \quad \forall t > 1$$

- **Decreasing returns to scale (DRS)** if:

$$f(tK, tL) < t \cdot f(K, L) \quad \forall t > 1$$

Common Production Functions

Cobb-Douglas:

$$f(K, L) = AK^\alpha L^\beta$$

where $A > 0$, $\alpha, \beta \in (0, 1)$

Leontief (Perfect Complements):

$$f(K, L) = \min\{aK, bL\}$$

CES (Constant Elasticity):

$$f(K, L) = A[\alpha K^\rho + (1 - \alpha)L^\rho]^{\frac{1}{\rho}}$$

Linear (Perfect Substitutes):

$$f(K, L) = aK + bL$$

Example: Cobb-Douglas Returns to Scale

Consider $f(K, L) = AK^\alpha L^\beta$

Scaling all inputs by $t > 0$:

$$\begin{aligned} f(tK, tL) &= A(tK)^\alpha (tL)^\beta \\ &= At^\alpha K^\alpha t^\beta L^\beta \\ &= t^{\alpha+\beta} AK^\alpha L^\beta \\ &= t^{\alpha+\beta} f(K, L) \end{aligned}$$

Therefore:

- If $\alpha + \beta = 1$: **CRS**
- If $\alpha + \beta > 1$: **IRS**
- If $\alpha + \beta < 1$: **DRS**

Cost Minimization

Cost minimization

Given input prices w and technology f , the firm chooses inputs z so as to produce output $\bar{q}$ at minimum cost.

Cost Minimization Problem

$$\begin{aligned} \min_z \quad & w \cdot z \\ \text{s.t.} \quad & f(z) = \bar{q} \end{aligned}$$

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Notice:

- The given parameters here are w and $\bar{q}$
- There is no output price p here

First-order Conditions

Set up the Lagrangian:

$$\mathcal{L} = w \cdot z + \lambda[\bar{q} - f(z)]$$

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$$\frac{\partial \mathcal{L}}{\partial z_i} = w_i - \lambda \frac{\partial f}{\partial z_i} = 0, \quad i = 1, \dots, I$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - f(z) = 0$$

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Combining the first-order conditions for z_i and z_j :

$$\frac{w_i}{w_j} = \frac{\partial f}{\partial z_i} \bigg/ \frac{\partial f}{\partial z_j} \equiv \text{MRTS}_{ij}$$

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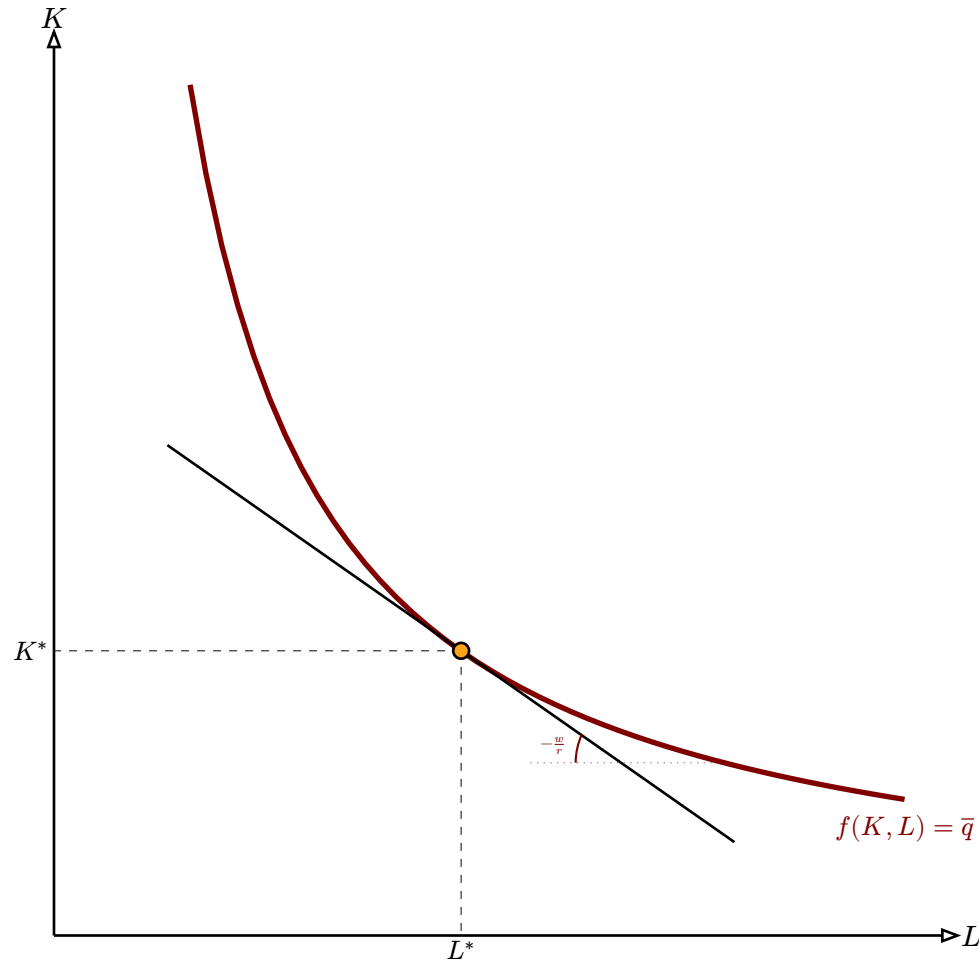
Treating w and $\bar{q}$ as given parameters, we can solve for z as a function of w and $\bar{q}$:

$$z_i^*(w, \bar{q})$$

This function is called **conditional factor demand** for z_i .

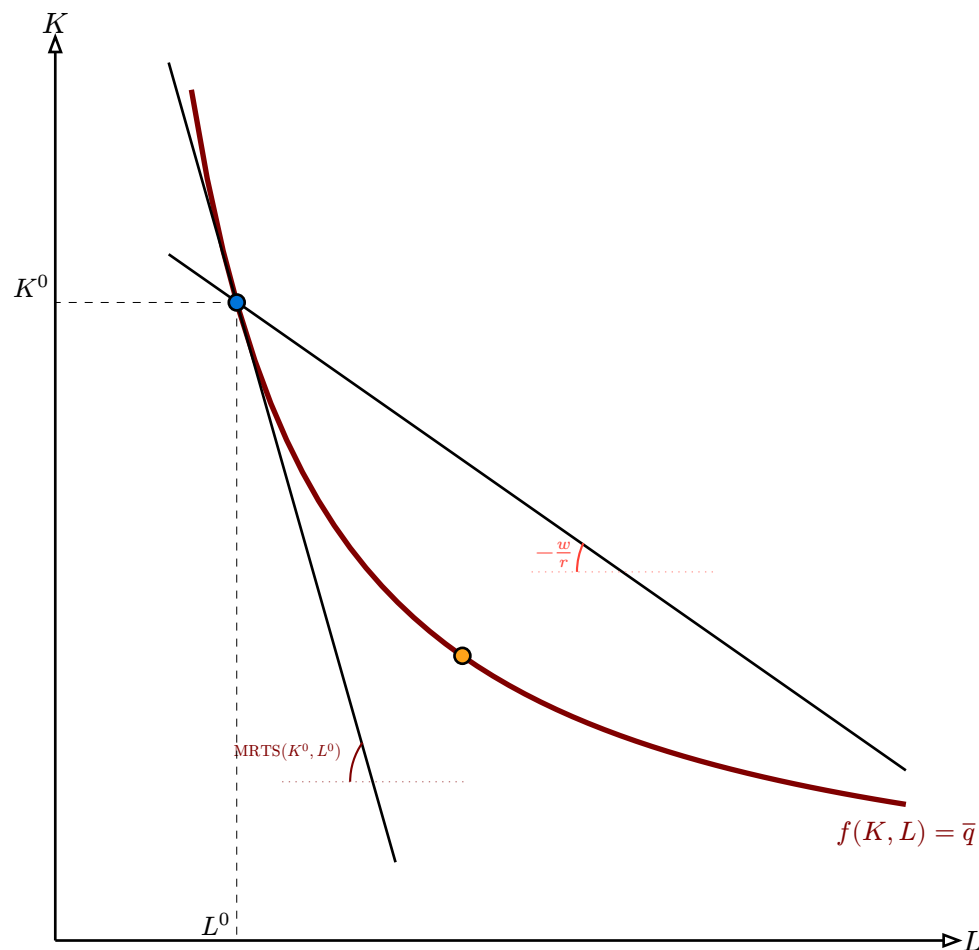
Explanation of the optimality condition

- The optimality condition is satisfied at (K^*, L^*) .

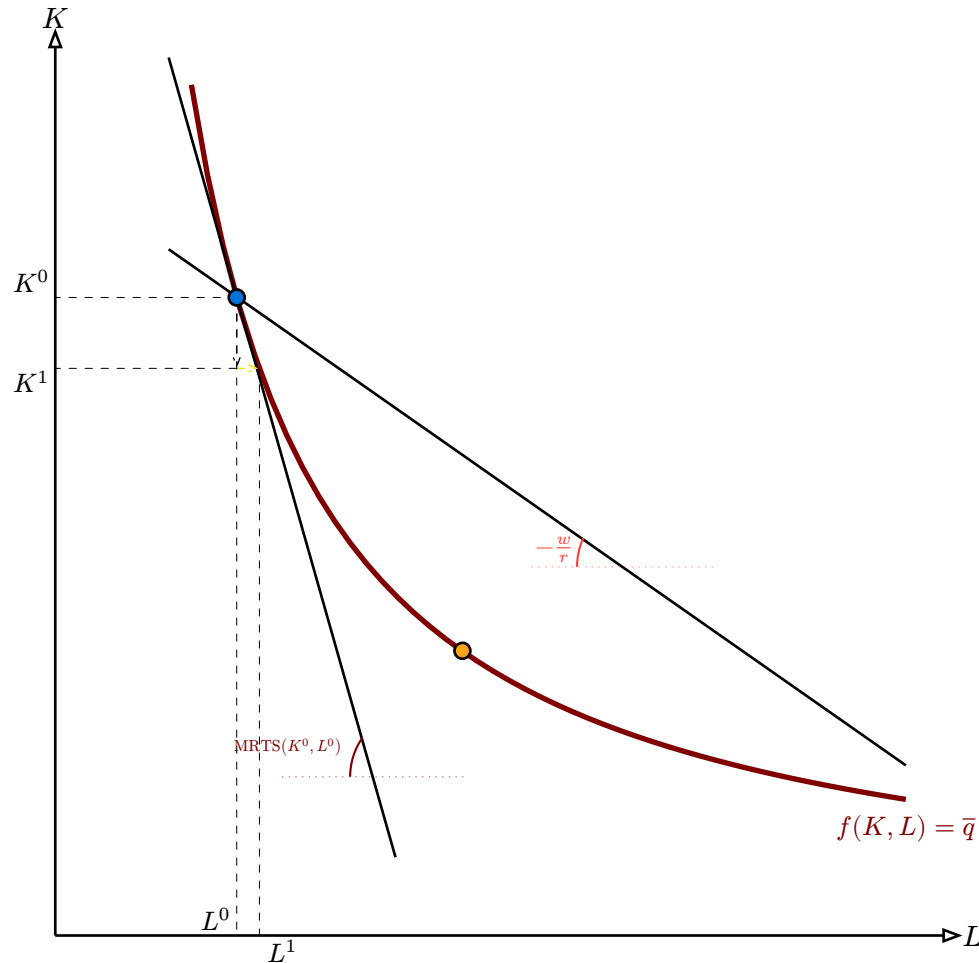


Explanation of the optimality condition

- Imagine that the firm produces at (K^0, L^0) , where $\frac{w}{r} \neq |\text{MRTS}|$

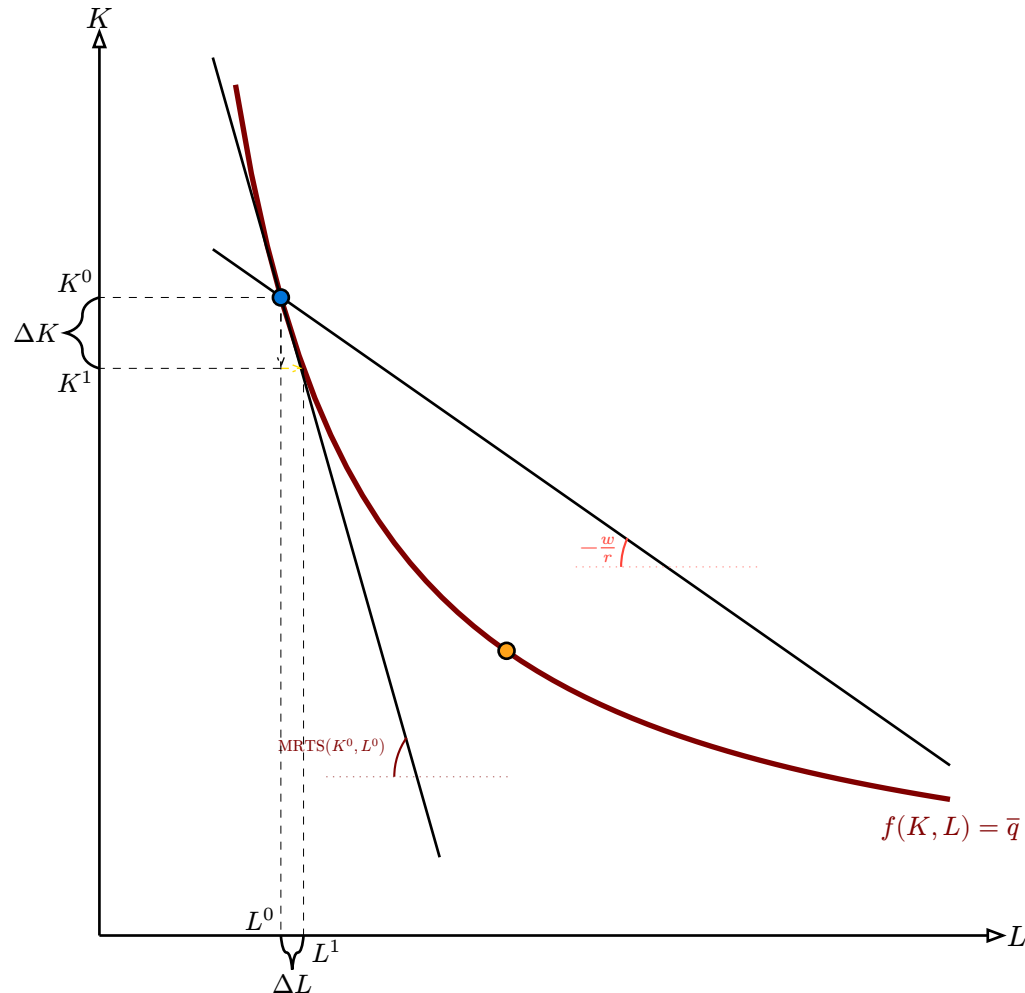


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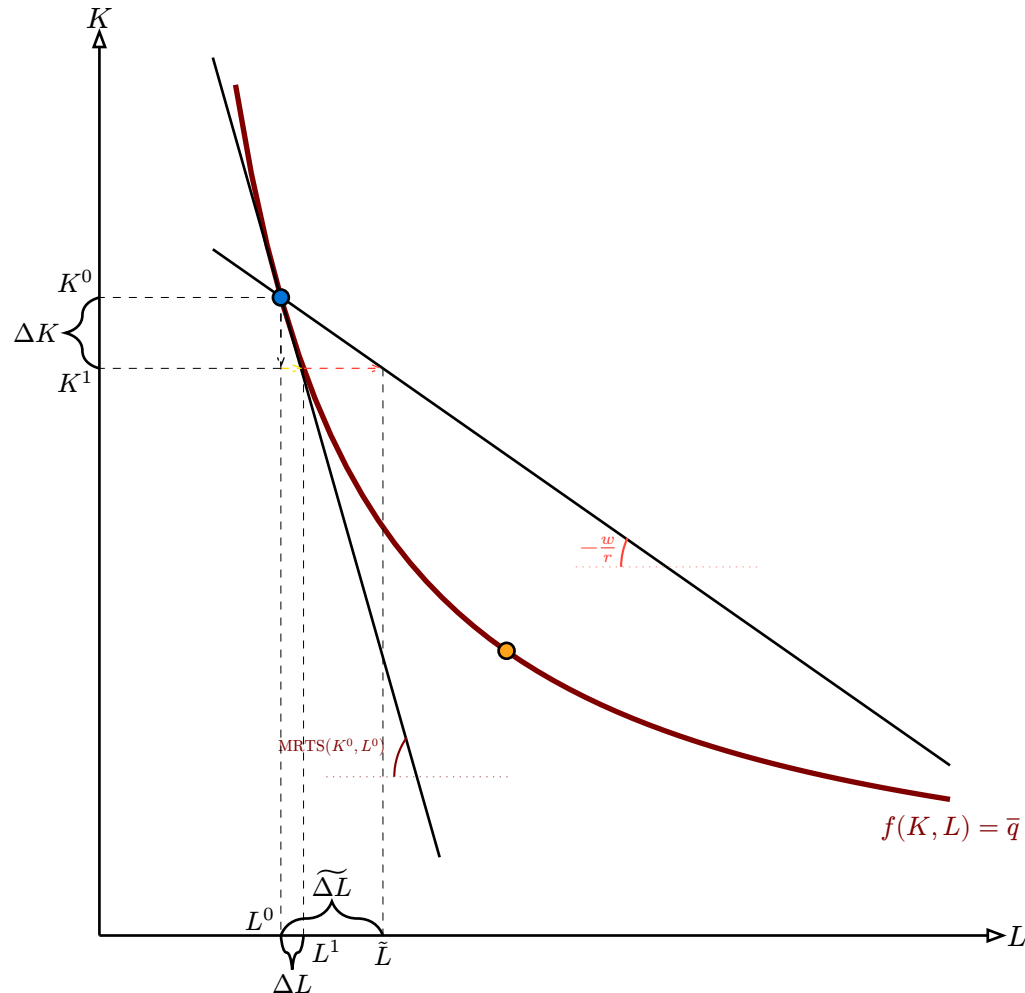
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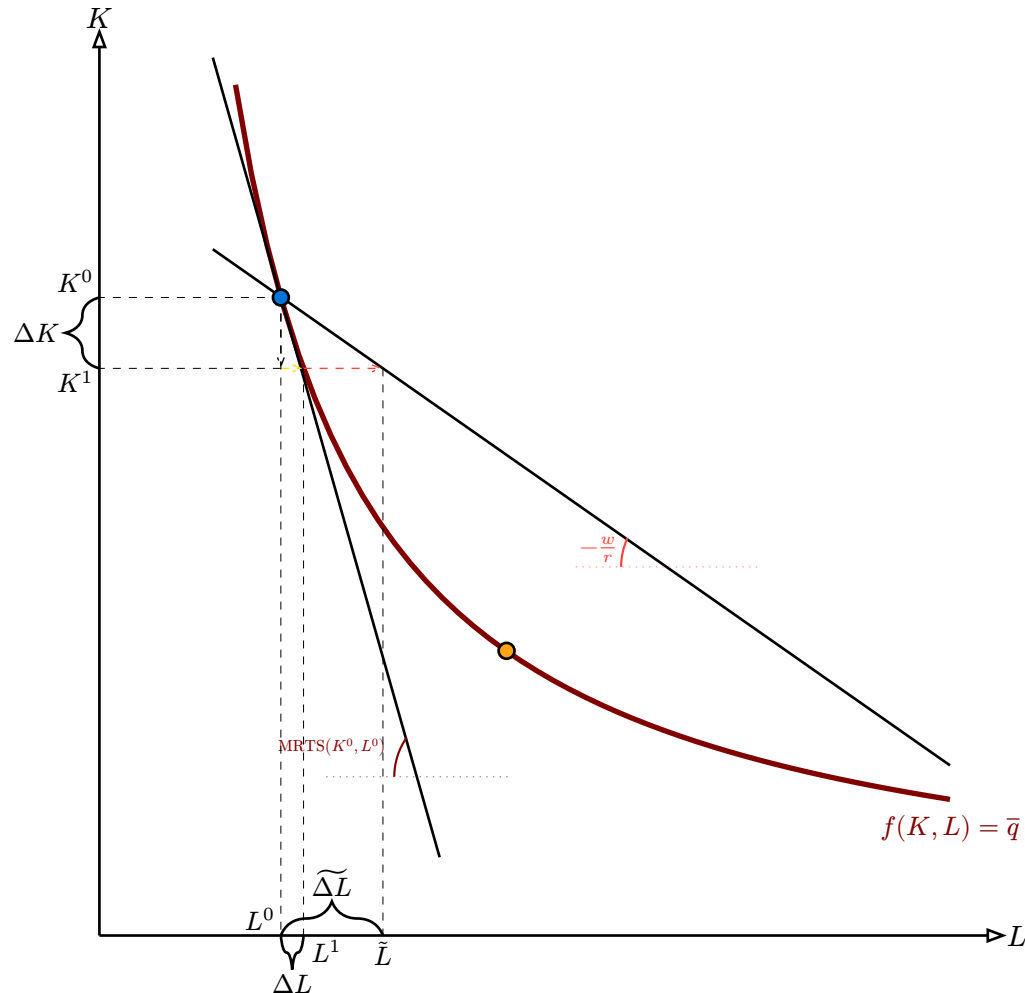
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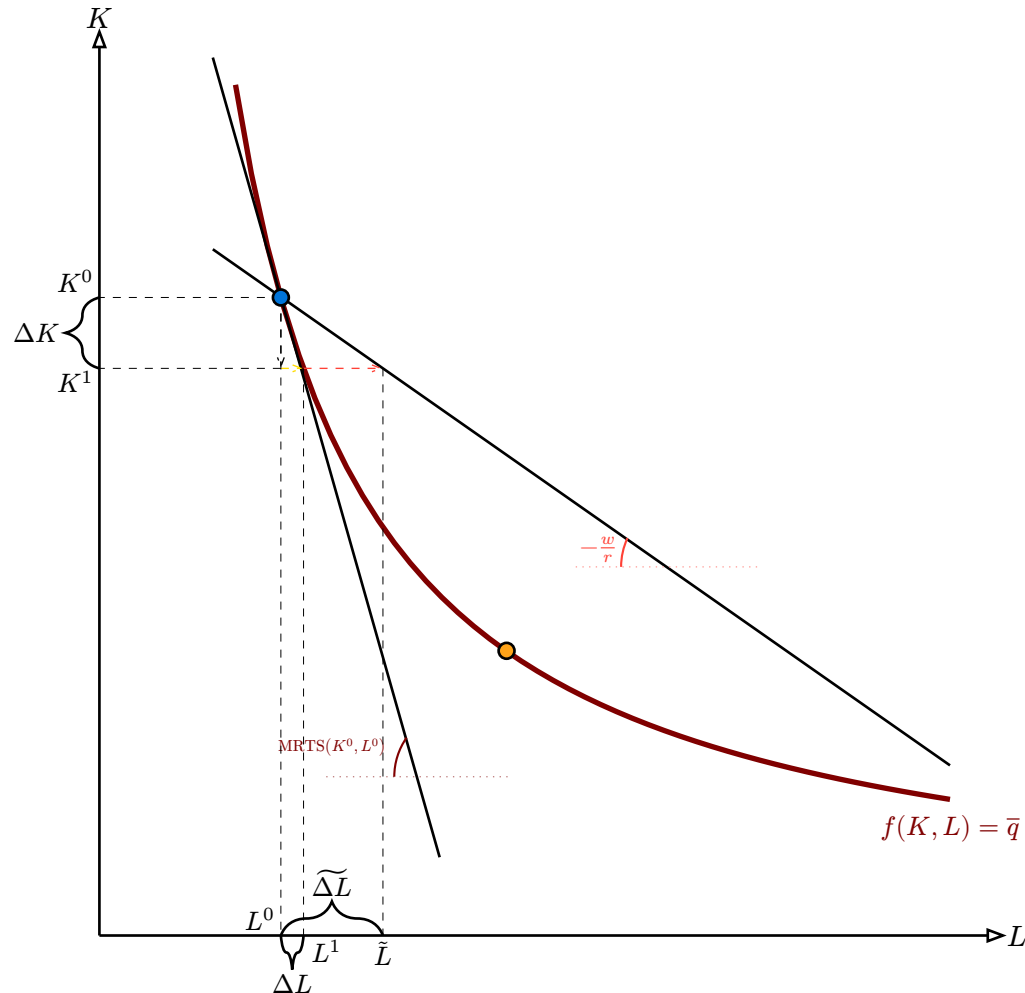
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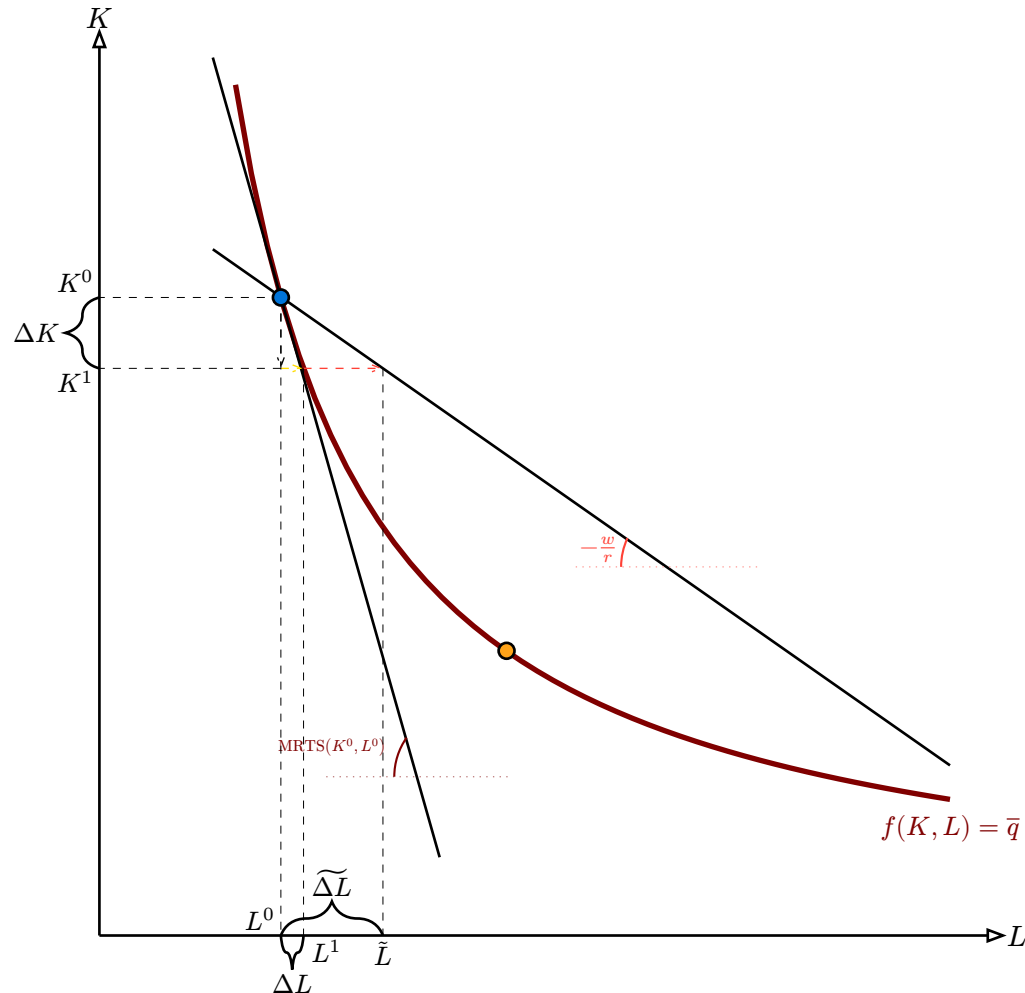
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- But for $r\Delta K$, the firm could get $\widetilde{\Delta L} > \Delta L$
- In other words, $r\Delta K > w\Delta L$
- Therefore, the firm can save $w(\widetilde{\Delta L} - \Delta L)$ and still achieve the target $\bar{q}$!
- Thus, the firm cannot minimize costs by using input mixes for which $\frac{w}{r} \neq |MRTS|$

Cost functions

Total cost function tells the production costs, conditional on the optimal choice of inputs:

$$c(w, \bar{q}) \equiv w \cdot z^*(w, \bar{q})$$

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Marginal cost function:

$$MC(w, \bar{q}) \equiv \frac{\partial c(w, \bar{q})}{\partial \bar{q}}$$

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Marginal cost function:

$$\begin{aligned} MC(w, \bar{q}) &\equiv \frac{\partial c(w, \bar{q})}{\partial \bar{q}} \\ &= \lambda^* \quad (\text{the optimal value of the Lagrange multiplier}) \end{aligned}$$

Wrap-up and Next Steps

Summary

This week we covered:

- ✓ Production functions and their key characteristics
- ✓ Returns to scale: CRS, IRS, DRS
- ✓ Marginal products and MRTS
- ✓ Cost minimization: firm chooses inputs to minimize cost
- ✓ Optimality condition: $MRTS = w/r$
- ✓ Conditional factor demand and cost functions

Next lecture: Cost curves and their properties